JOURNAL OF APPROXIMATION THEORY **84**, 247–264 (1996)
ARTICLE NO. 0018

# On Total Positivity of the Discrete Spline Collocation Matrix

KNUT M. MØRKEN*

*Department of Informatics, University of Oslo, Box 1080, Blindern, 0316 Oslo, Norway*

*Communicated by Charles A. Micchelli*

Received September 24, 1993; accepted in revised form June 6, 1995

R. Q. Jia (*J. Approx. Theory* **39** (1983), 11–23), proved that the discrete spline collocation matrix was totally positive, and he also gave necessary and sufficient conditions for when a minor has a positive determinant. By a counterexample, it is shown in this paper that his necessary and sufficient conditions are not quite correct. The correct form of the theorem is then established. 

## 1. Introduction and Preliminary Results

In Theorem 1 in [2], Jia proved that the discrete B-spline collocation matrix is totally positive, and he also gave necessary and sufficient conditions for a minor to be strictly positive. However, these conditions are not quite correct. In this paper, we first give a counterexample to Jia's result and then establish the correct necessary and sufficient conditions for positivity of a minor.

In the rest of this section we introduce the concepts that are necessary to study the total positivity of discrete B-splines. We then state Jia's result and give a counterexample before we present the correct version of the theorem. Section 2 is devoted to the proof of the theorem, and in Section 3 we give some consequences and applications of the result.

Discrete B-splines can be defined and studied in their own right, but the main motivation comes from traditional spline theory. Let $k$ be a positive integer, and let $\mathbf{t}=(t_i)_{i=-\infty}^{\infty}$ be a bi-infinite sequence of real numbers with $t_i<t_{i+k}$ for all $i$. (We will only use finite parts of the knot vectors; the assumption of bi-infinity is only for notational convenience.) We can then associate the usual polynomial B-splines $\{B_{i,k,t}\}_{i=-\infty}^{\infty}$ with $\mathbf{t}$, right continuous and normalized to sum to one. It is well known that these B-splines provide a basis for the linear space $\mathbb{S}_{k,t}$ of piecewise polynomials of order $k$ with joins at the knots in $\mathbf{t}$ (it $t_z$ occurs $m$ times in $i$, then an element $f$

* E-mail: knutm@ifi.uio.no.



0021-9045/96 $18.00

of $\mathbb{S}_{k,t}$ will have $k-m-1$ continuous derivatives in a neighbourhood of $t_z$, but the derivative of order $k-m$ may be discontinuous).

To introduce discrete B-splines, let $\boldsymbol{\tau}$ be a bi-infinite subsequence of $\mathbf{t}$. We then have $\mathbb{S}_{k,\boldsymbol{\tau}} \subseteq \mathbb{S}_{k,\mathbf{t}}$, so that any B-spline $B_{j,k,\boldsymbol{\tau}}$ associated with $\boldsymbol{\tau}$ can be written as a linear combination of the B-splines in $\mathbb{S}_{k,\mathbf{t}}$,

$$B_{j,k,\boldsymbol{\tau}} = \sum_i \alpha_{j,k,\boldsymbol{\tau},\mathbf{t}}(i)\, B_{i,k,\mathbf{t}}. \tag{1}$$

The functions $\{\alpha_{j,k,\boldsymbol{\tau},\mathbf{t}}\}_j$, defined on the integers $\mathbb{Z}$, are called discrete B-splines (we will often shorten $\alpha_{j,k,\boldsymbol{\tau},\mathbf{t}}(i)$ by omitting some of the subscripts when there is no chance of ambiguity). Discrete B-splines have many properties similar to B-splines; see [2, 3]. We mention in particular that they obey a recurrence relation very similar to the recurrence relation for B-splines, and this recurrence leads to stable algorithms for computing discrete B-splines. Given the coefficients $(c_j)$ of a spline relative to the B-splines in $\mathbb{S}_{k,\boldsymbol{\tau}}$, we can compute its coefficients $(d_i)$ relative to the B-splines in $\mathbb{S}_{k,\mathbf{t}}$ via the formula $d_i = \sum_j c_j \alpha_j(i)$, which is immediate from (1). This process of refining the knot vector is often called knot insertion or subdivision, and is of fundamental importance in most applications of splines.

In this paper we are going to study the discrete B-spline collocation matrix, $A_{\boldsymbol{\tau},\mathbf{t}}$ (often shortened to $A$) with elements given by $(A)_{i,j} = \alpha_j(i)$, i.e.,

$$A = \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \cdots \\ \cdots & \alpha_{-1}(-1) & \alpha_0(-1) & \alpha_1(-1) & \cdots \\ \cdots & \alpha_{-1}(0) & \alpha_0(0) & \alpha_1(0) & \cdots \\ \cdots & \alpha_{-1}(1) & \alpha_0(1) & \alpha_1(1) & \cdots \\ \cdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{2}$$

That $A$ is totally positive means that for any two increasing integer sequences $i_1 < i_2 < \cdots < i_m$ and $j_1 < j_2 < \cdots < j_m$ of length $m \geqslant 1$, the submatrix of $A$ given by

$$A\begin{bmatrix} i_1, ..., i_m \\ j_1, ..., j_m \end{bmatrix} = \begin{pmatrix} \alpha_{j_1}(i_1) & \alpha_{j_2}(i_1) & \cdots & \alpha_{j_m}(i_1) \\ \alpha_{j_1}(i_2) & \alpha_{j_2}(i_2) & \cdots & \alpha_{j_m}(i_2) \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{j_1}(i_m) & \alpha_{j_2}(i_m) & \cdots & \alpha_{j_m}(i_m) \end{pmatrix}$$

has nonnegative determinant,

$$\det A\begin{bmatrix} i_1, ..., i_m \\ j_1, ..., j_m \end{bmatrix} \geqslant 0.$$

The proof of this is quite straightforward; the challenge is to determine exactly when the determinant is strictly positive.

In special cases the value of a B-spline and its derivatives can be obtained from discrete B-splines. We will use this in the last section of the paper to deduce the Schoenberg–Whitney theorem for B-splines and related results from the total positivity result for discrete B-splines. In this respect, the following lemma will be useful.

LEMMA 1. *Suppose that $k \geqslant 2$ and that $\mathbf{t}$ is a knot vector. Suppose also that the knot $t_i$ satisfies $t_i \leqslant t_{i+1} = \cdots = t_{i+k-r-1} < t_{i+k-r}$ for some $r<k$, and that $a = t_{i+1}$. If $f = \sum_j c_j B_{j,k,\mathbf{t}}$, then*

$$f^{(r)}(a) = D_{k-r} D_{k-r+1} \cdots D_{k-1} c_i, \tag{3}$$

*where*

$$D_n d_j = \begin{cases} (d_j - d_{j-1})/((t_{j+n} - t_j)/n), & \text{if } \quad t_j < t_{j+n}; \\ 0, & \text{otherwise}; \end{cases}$$

*for $n = k-1, k-2, ..., k-r$. In particular, if $\boldsymbol{\tau}$ is a subsequence of $\mathbf{t}$, then*

$$B^{(r)}_{j,k,\boldsymbol{\tau}}(a) = D_{k-r} D_{k-r+1} \cdots D_{k-1} \alpha_{j,k}(i). \tag{4}$$

*Proof.* By elementary properties of B-splines, we have that $B_{j,k-r,\mathbf{t}}(a) = \delta_{i,j}$ for fixed $a = t_{i+1}$. On the other hand, from the well-known differentiation formula for B-splines, we have

$$f^{(r)}(x) = \sum_j (D_{k-r} \cdots D_{k-2} D_{k-1} c_j)\, B_{j,k-r,\mathbf{t}}(x).$$

From this (3) follows. The relation (4) results when (3) is applied to (1). ∎

As in discussions of many other properties of discrete B-splines, we need some simple functions that count various multiplicities of knots.

DEFINITION 2. The expressions $m_{\mathbf{t}}$, $l_{\mathbf{t}}(i)$, and $r_{\mathbf{t}}(i)$ denote the total number of occurrences of the real number $x$ in the knot vector $\mathbf{t}$; the number of knots in $\mathbf{t}$ equal to $t_i$, but with index less than $i$; and the number of knots in $\mathbf{t}$ equal to $t_i$, but with index greater than $i$; respectively. More formally, we have

$$\begin{aligned} m_{\mathbf{t}}(x) &= \max\{q - p \mid t_q \leqslant x \text{ and } x \leqslant t_{p+1}\} \\ l_{\mathbf{t}}(i) &= \max\{p \mid t_{i-p} = t_i\} \\ r_{\mathbf{t}}(i) &= \max\{p \mid t_{i+p} = t_i\}. \end{aligned}$$

These quantities are called the multiplicity of $x$ in $\mathbf{t}$, the left multiplicity of $t_i$ in $\mathbf{t}$, and the right multiplicity of $t_i$ in $\mathbf{t}$, respectively.

Note that with this notation we have the relations

$$t_{i-l_{\mathbf{t}}(i)-1} < t_{i-l_{\mathbf{t}}(i)} = t_i = t_{i+r_{\mathbf{t}}(i)} < t_{i+r_{\mathbf{t}}(i)+1}, \tag{5}$$

which can be verified directly.

With the multiplicity counting functions we can also conveniently state some results which characterize the support properties of discrete B-splines which we will need later. We single out the special cases where there are none or only one more knot in $\mathbf{t}$ than in $\boldsymbol{\tau}$. For proofs, see, for example, [2]. The phrase "The knot vector $\boldsymbol{\tau}$ is formed by dropping an entry $t_z$ from $\boldsymbol{\tau}$" is used repeatedly below. The precise meaning of this is that $\tau_i = t_i$ for $i < z$ and $\tau_i = t_{i+1}$ for $i \geqslant z$.

LEMMA 3. *If the knot vectors $\boldsymbol{\tau}$ and $\mathbf{t}$ are identical, then*

$$\alpha_j(i) = \delta_{i,j} = \begin{cases} 1, & \text{if} \quad i = j; \\ 0, & \text{otherwise.} \end{cases}$$

*If the knot vector $\boldsymbol{\tau}$ is formed by dropping an entry $t_z$ from $\mathbf{t}$, then*

$$\begin{aligned} \alpha_j(i) &= 0, && \text{for} \quad j < i-1 \ \text{or} \ j > i; \\ \alpha_{i-1}(i) &\geqslant 0, && \text{with strict inequality iff } t_{i+k} > t_z; \\ \alpha_i(i) &\geqslant 0, && \text{with strict inequality iff } t_i < t_z. \end{aligned}$$

LEMMA 4. *Let $\mathbf{t}$ and $\boldsymbol{\tau}$ be knot vectors for splines of order $k$ such that $\boldsymbol{\tau}$ is a subsequence of $\mathbf{t}$. Then*

$$\alpha_j(i) \geqslant 0,$$

*with equality if and only if one of the following four cases occurs:*

(i) $t_i < \tau_j$;

(ii) $t_i = \tau_j$ *and* $r_{\mathbf{t}}(i) > r_{\boldsymbol{\tau}}(j)$;

(iii) $t_{i+k} > \tau_{j+k}$;

(iv) $t_{i+k} = \tau_{j+k}$ *and* $l_{\boldsymbol{\tau}}(j+k) < l_{\mathbf{t}}(i+k)$.

We will need one more property of knot insertion. This property is simple, but powerful, and basically says that the order in which knots are inserted is irrelevant; see [2].

LEMMA 5. *Suppose that* $\boldsymbol{\rho}$ *is a subsequence of* $\mathbf{t}$ *and that* $\boldsymbol{\tau}$ *is a subsequence of* $\boldsymbol{\rho}$. *Then*

$$\alpha_{j,k,\boldsymbol{\tau},\mathbf{t}}(i)=\sum_{l}\alpha_{l,k,\boldsymbol{\rho},\mathbf{t}}(i)\,\alpha_{j,k,\boldsymbol{\tau},\boldsymbol{\rho}}(l),$$

$$A_{\boldsymbol{\tau},\mathbf{t}}=A_{\boldsymbol{\rho},\mathbf{t}}A_{\boldsymbol{\tau},\boldsymbol{\rho}},$$

*and*

$$\det A_{\boldsymbol{\tau},\mathbf{t}}\begin{bmatrix} i_1,\dots,i_m \\ j_1,\dots,j_m \end{bmatrix}=\sum_{\substack{\xi_1<\cdots<\xi_m \\ (\xi_i)_{i=1}^m\in\mathbb{Z}^m}}\det A_{\boldsymbol{\rho},\mathbf{t}}\begin{bmatrix} i_1,\dots,i_m \\ \xi_1,\dots,\xi_m \end{bmatrix}\det A_{\boldsymbol{\tau},\boldsymbol{\rho}}\begin{bmatrix} \xi_1,\dots,\xi_m \\ j_1,\dots,j_m \end{bmatrix}.$$

## 2. A COUNTEREXAMPLE AND THE CORRECTED RESULT

Let us now turn to Jia's theorem 1. We first state the result and give a counterexample and then give a corrected version. With our notation, Theorem 1 of [2] can be expressed as follows.

(JIA [2]). *Let* $k$ *be a positive integer, let* $\mathbf{t}$ *be a knot vector with* $t_i<t_{i+k}$ *for all* $i$, *and let* $\boldsymbol{\tau}$ *be a subsequence of* $\mathbf{t}$. *Let* $i_1<i_2<\cdots<i_m$ *and* $j_1<j_2<\cdots<j_m$ *be two increasing integer sequences. Then*

$$\det A\begin{bmatrix} i_1,\dots,i_m \\ j_1,\dots,j_m \end{bmatrix}\geqslant 0,$$

*with strict positivity if and only if both of the following conditions are satisfied*:

(i) $\alpha_{j_q}(i_q)>0$ *for* $q=1,2,\dots,m$.

(ii) *If for some* $q$, *the multiplicity of* $t_{i_q}$ *in* $\mathbf{t}$ *is greater than the multiplicity of* $t_{i_q}$ *in* $\boldsymbol{\tau}$, *that is,* $m_{\boldsymbol{\tau}}(t_{i_q})<m_{\mathbf{t}}(t_{i_q})$, *then*

$$i_{q-d_q}<i_q-d_q, \tag{6}$$

*where* $d_q=k-r_{\mathbf{t}}(i_q)$.

(In [2], the inequality $m_{\boldsymbol{\tau}}(t_{i_q})<m_{\mathbf{t}}(t_{i_q})$ has been replaced by the equality $m_{\boldsymbol{\tau}}(t_{i_q})=m_{\mathbf{t}}(t_{i_q})$. This is a misprint as is apparent from studying the "proof" of the theorem.)

It is condition (ii) of the theorem that is not quite correct. This condition turns out to be too weak to assure positivity of the determinant in all cases. Consider quadratic splines ($k=3$) on the knot vectors

$$\boldsymbol{\tau}=(0, 0, 0, 3, 3, 3), \qquad \mathbf{t}=(0, 0, 0, 2, 2, 2, 3, 3, 3).$$

Then it is a simple matter to compute the collocation matrix $A$ as

$$A=\begin{pmatrix} 1 & 0 & 0 \\ 1/3 & 2/3 & 0 \\ 1/9 & 4/9 & 4/9 \\ 1/9 & 4/9 & 4/9 \\ 0 & 1/3 & 2/3 \\ 0 & 0 & 1 \end{pmatrix}.$$

From this it is immediately evident that, unlike in the case of continuous splines, it is not sufficient for positivity of a minor that the diagonal is positive, since rows 3 and 4 are equal. The purpose of condition (ii) in Jia's result is to ensure that such linear dependencies do not occur in

$$V=A\begin{bmatrix} i_1, \ldots, i_m \\ j_1, \ldots, j_m \end{bmatrix}.$$

In this particular case we see that if $i_q=4$, then we have $d_q=1$, and the inequality (6) reads $i_{q-1}<i_q-1=3$. This inequality therefore makes sure that row 3 of $A$ does not occur together with row 4, in a positive minor.

Let us now turn to a counterexample. Choose $i_1=2$, $i_2=3$, $i_3=5$ and $j_1=1$, $j_2=2$, $j_3=3$. The submatrix is then

$$V=A\begin{bmatrix} i_1, i_2, i_3 \\ j_1, j_2, j_3 \end{bmatrix}=\begin{pmatrix} 1/3 & 2/3 & 0 \\ 1/9 & 4/9 & 4/9 \\ 0 & 1/3 & 2/3 \end{pmatrix}.$$

We observe that condition (i) of Jia's result is satisfied. Moreover, since $m_{\mathbf{t}}(t_{i_1})=m_{\mathbf{t}}(t_{i_2})=m_{\boldsymbol{\tau}}(t_{i_1})=m_{\boldsymbol{\tau}}(t_{i_2})$, condition (ii) only applies to $i_3$. Here we find $r_{\mathbf{t}}(i_3)=1$ and $d_3=3-1=2$, so that the condition is

$$2=i_1<i_3-2=3,$$

which is certainly satisfied. Therefore both conditions (i) and (ii) are satisfied, but the determinant is still 0, since $v_2=(v_1+2v_3)/3$, where $v_1$, $v_2$, and $v_3$ denote the three rows of $V$.

The correct version of the theorem is slightly more complicated than Jia's statement.

THEOREM 6. *Let $k$ be a positive integer, let $\mathbf{t}$ be a knot vector with $t_i < t_{i+k}$ for all $i$, and let $\boldsymbol{\tau}$ be a subsequence of $\mathbf{t}$. Let $i_1 < i_2 < \cdots < i_m$ and $j_1 < j_2 < \cdots < j_m$ be two increasing integer sequences. Then*

$$\det V = \det A \begin{bmatrix} i_1, ..., i_m \\ j_1, ..., j_m \end{bmatrix} \geqslant 0,$$

*with strict positivity if and only if both of the following conditions are satisfied*:

(i) *The diagonal of $V$ is positive, i.e.,*

$$\alpha_{j_q}(i_q) > 0 \qquad \textit{for} \quad q = 1, 2, ..., m.$$

(ii) *If for some $q$, the multiplicity of $t_{i_q}$ in $\mathbf{t}$ is greater than the multiplicity of $t_{i_q}$ in $\boldsymbol{\tau}$, that is, $m_{\boldsymbol{\tau}}(t_{i_q}) < m_{\mathbf{t}}(t_{i_q})$, then*

$$i_{q-d_q} < i_q - d_q - f_q,$$

*where*

$$d_q = k - r_{\mathbf{t}}(i_q)$$

*and*

$$f_q = \min\{l_{\mathbf{t}}(i_q), m_{\mathbf{t}}(t_{i_q}) - m_{\boldsymbol{\tau}}(t_{i_q}) - 1\}.$$

If $l_{\mathbf{t}}(i_q) = 0$ or $m_{\mathbf{t}}(t_{i_q}) - m_{\boldsymbol{\tau}}(t_{i_q}) = 1$ for all $q$ such that $m_{\boldsymbol{\tau}}(t_{i_q}) < m_{\mathbf{t}}(t_{i_q})$, then condition (ii) in Jia's result and Theorem 6 are identical. Note also that if the sequence $(i_q)$ is too short for $i_{q-d_q}$ to be defined, then condition (ii) of Theorem 6 is automatically satisfied.

## 3. A PROOF OF THE CORRECTED RESULT

In the rest of the paper, we will refer back to Theorem 6 many times, especially the two conditions for positivity of the minor. Let us name these as $C1$ and $C2$.

In Theorem 6, we have introduced a new parameter $f_q$ that fits into the relations in (5),

$$t_{i-l_{\mathbf{t}}(i)-1} < t_{i-l_{\mathbf{t}}(i)} = t_i = t_{i+r_{\mathbf{t}}(i)-f_q} = t_{i+r_{\mathbf{t}}(i)} < t_{i+r_{\mathbf{t}}(i)+1}. \tag{7}$$

These relations will be used many times in the proof of Theorem 6. Another simple observation that will be important is that if $(i_q)_q$ is a nondecreasing sequence then $i_{q-p} \leqslant i_q - p$ for all positive integers $p$ such that $i_{q-p}$ is defined.

Before entering into the proof, let us identify some special cases where $C2$ is redundant. This will reveal more of the significance of this condition and also be useful in the proof of Theorem 6.

COROLLARY 7. *In the following two special cases, $C1$ of Theorem 6 implies $C2$:*

(a) *There is only one more knot in $\mathbf{t}$ than in $\boldsymbol{\tau}$.*

(b) *For all $q$ such that $m_{\boldsymbol{\tau}}(t_{i_q}) < m_{\mathbf{t}}(t_{i_q})$, the number $i_q$ satisfies $t_{i_q} < t_{i_q+1}$.*

*Proof.* Let us start with case (a). Let $t_z$ be the new knot and suppose that $t_{i_q} = t_z$ and that the diagonal is positive. Then $\alpha_{j_q}(i_q) > 0$, so that by Lemma 3, we must have

$$j_q = i_q - 1. \tag{8}$$

Suppose that $C2$ does not hold, i.e., that $i_{q-d_q} = i_q - d_q$. From the definition of $d_q$ in Theorem 6 and from Definition 2, we see that this is equivalent to $i_{q-d_q} + k = i_q + r_{\mathbf{t}}(i_q)$, so that $t_{i_{q-d_q}+k} = t_z$; see (7). Since $\alpha_{j_{q-d_q}}(i_{q-d_q}) > 0$, this, combined with Lemma 3, means that

$$j_{q-d_q} = i_{q-d_q} = i_q - d_q. \tag{9}$$

But (8) and (9) now give $j_q - j_{q-d_q} = d_q - 1$ which is impossible since the sequence $(j_q)_q$ is strictly increasing.

In case (b), we see that we must have $r_{\mathbf{t}}(i_q) = 0$ and therefore $d_q = k$. Assuming again that the diagonal is positive, we deduce from $\alpha_{j_q}(i_q) > 0$ and Lemma 4 that

$$\tau_{j_q} \leqslant t_{i_q}. \tag{10}$$

On the other hand, if $C2$ does not hold and $m_{\boldsymbol{\tau}}(t_{i_q}) < m_{\mathbf{t}}(t_{i_q})$, we have $i_{q-k} \geqslant i_q - k - f_q$ or $i_{q-k} + k \geqslant i_q - f_q$. The inequality $\alpha_{j_{q-k}}(i_{q-k}) > 0$, combined with Lemma 4, gives $\tau_{j_{q-k}+k} > t_{i_q}$ (it is not possible to have $\tau_{j_{q-k}+k} = t_{i_q}$ since we would then need $l_{\boldsymbol{\tau}}(j_{q-k}+k) \geqslant l_{\mathbf{t}}(i_{q-k}+k) \geqslant l_{\mathbf{t}}(i_q - f_q) = m_{\boldsymbol{\tau}}(t_{i_q})$). Since the sequence $(j_q)_q$ is strictly increasing, we must have $j_q \geqslant j_{q-k} + k$, and therefore $\tau_{j_q} > t_{i_q}$ which contradicts (10). ∎

In order to prove Theorem 6, we follow an approach similar to that in [2]. The fact that all minors of $A$ are nonnegative is a consequence of Lemma 5 (it is simple to check if there is only one more knot in $\mathbf{t}$ than in $\boldsymbol{\tau}$, and the general case then follows by induction). The difficult part is to determine exactly when a minor is positive. For convenience, we list the different steps as lemmas. We first show that if $C1$ is violated, the submatrix of $A$ that we are considering,

$$V = A\begin{bmatrix} i_1, \dots, i_m \\ j_1, \dots, j_m \end{bmatrix},$$

is singular.

LEMMA 8. *If $C1$ does not hold, then $V$ is singular.*

*Proof.* This follows from standard arguments (see [2]), but for completeness we reproduce the proof here. Let $\alpha_{j_q}(i_q)$ be a zero diagonal entry in $V$. Then we must either have

$$\alpha_{j_r}(i_s) = 0 \qquad \text{for} \quad 1 \leqslant r \leqslant q \leqslant s \leqslant m,$$

or

$$\alpha_{j_r}(i_s) = 0 \qquad \text{for} \quad 1 \leqslant s \leqslant q \leqslant r \leqslant m.$$

In the first case, the $m-q+1$ last rows of $V$ must be linearly dependent since only the last $m-q$ entries of each row can be nonzero. Similarly, in the second case, the first $q$ rows of $V$ must be linearly dependent. ∎

LEMMA 9. *If $C2$ does not hold, then $V$ is singular.*

*Proof.* Let $\mathbf{s}$ be the sequence of knots that are in $\mathbf{t}$ but not in $\boldsymbol{\tau}$, i.e., all distinct $x$ in $\mathbf{t}$ occur $m_{\mathbf{t}}(x) - m_{\boldsymbol{\tau}}(x)$ times in $\mathbf{s}$. The knots in $\mathbf{s}$ can conveniently be called the new knots. The proof is by induction on $\#\mathbf{s}$, the length of the sequence $\mathbf{s}$ (the number of new knots).

If $\#\mathbf{s} = 0$, then $\boldsymbol{\tau} = \mathbf{t}$, so that $C2$ never applies.

If $\#\mathbf{s} = 1$, then case (a) in Corollary 7 shows that if $C2$ does not hold then neither does $C1$, so that by Lemma 8 we find that $V$ must be singular.

Suppose now that $\#\mathbf{s} \geqslant 2$ and that we have established the result for knot vectors where the number of new knots is smaller than $\#\mathbf{s}$. Let $t_{i_q} = t_z$ be such that $m_{\boldsymbol{\tau}}(t_z) < m_{\mathbf{t}}(t_z)$ and $i_{q-d_q} \geqslant i_q - d_q - f_q$. Form the knot vector $\boldsymbol{\rho}$ by dropping $t_z$ from $\mathbf{t}$. Then we know from Lemma 5 that

$$\det A_{\boldsymbol{\tau},\mathbf{t}}\begin{bmatrix} i_1, \dots, i_m \\ j_1, \dots, j_m \end{bmatrix} = \sum_{\xi_1 < \cdots < \xi_m} \det A_{\boldsymbol{\rho},\mathbf{t}}\begin{bmatrix} i_1, \dots, i_m \\ \xi_1, \dots, \xi_m \end{bmatrix} \det A_{\boldsymbol{\tau},\boldsymbol{\rho}}\begin{bmatrix} \xi_1, \dots, \xi_m \\ j_1, \dots, j_m \end{bmatrix}.$$

Let us for simplicity set

$$B_{\boldsymbol{\xi}} = A_{\boldsymbol{\tau},\mathbf{t}}\begin{bmatrix} i_1, \dots, i_m \\ \xi_1, \dots, \xi_m \end{bmatrix}, \qquad C_{\boldsymbol{\xi}} = A_{\boldsymbol{\tau},\boldsymbol{\rho}}\begin{bmatrix} \xi_1, \dots, \xi_m \\ j_1, \dots, j_m \end{bmatrix}.$$

Assuming the induction hypothesis holds, we will show that for all strictly increasing integer sequences $\xi$ we either have $\det B_\xi = 0$ or $\det C_\xi = 0$.

From Lemma 3 and Lemma 8 we know that if $\xi_q \neq i_q - 1$, then $\alpha_{\xi_q, k, \boldsymbol{\rho}, \mathbf{t}}(i_q) = 0$ so that $\det B_\xi = 0$. We therefore only need to check the case $\xi_q = i_q - 1$. Note that we always have $i_{q-d_q} \leqslant i_q - d_q$ so that $i_{q-d_q} + k \leqslant i_q + r_{\mathbf{t}}(i_q)$. From (7) we therefore conclude that $t_{i_{q-dq}+k} \leqslant t_z$, and Lemma 3 then shows that $\det B_\xi = 0$ unless $\xi_{q-d_q} = i_{q-d_q}$.

To complete the proof, it suffices to show that if $\xi_q = i_q - 1$ and $\xi_{q-d_q} = i_{q-d_q}$, then $\det C_\xi = 0$. Denote by $\tilde{d}_q$ and $\tilde{f}_q$ the parameters of $C2$ for the discrete B-splines with respect to the knot vectors $\boldsymbol{\tau}$ and $\boldsymbol{\rho}$. Then we see that $\tilde{d}_q = d_q$ since $r_{\mathbf{t}}(i_q) = r_{\boldsymbol{\rho}}(\xi_q)$. Therefore, if $f_q = 0$ we have

$$\xi_q = i_q - 1, \qquad \xi_{q-d_q} = i_{q-d_q} = i_q - d_q = \xi_q - d_q + 1,$$

which contradicts the fact that $(\xi_q)_q$ is a strictly increasing sequence.

If $f_q > 0$ we will show that $C2$ cannot hold for $C_\xi$. Since $f_q > 0$ we have $l_{\mathbf{t}}(i_q) > 0$, and, hence, $\rho_{\xi_q} = t_{i_q}$. Since in addition $\xi_q = i_q - 1$, we conclude that $\tilde{f}_q = f_q - 1$. We therefore find

$$\xi_{q-\tilde{d}_q} = \xi_{q-d_q} = i_{q-d_q} \geqslant i_q - d_q - f_q = \xi_q - d_q - (f_q - 1) = \xi_q - \tilde{d}_q - \tilde{f}_q.$$

Thus, we see that $C2$ is not satisfied on the reduced knot vector $\boldsymbol{\rho}$, and therefore, by the induction hypothesis, we have $\det C_\xi = 0$. ∎

The final step in the proof of Theorem 6 is to show that if both Conditions (i) and (ii) hold then the matrix is nonsingular.

LEMMA 10. *If* $C1$ *and* $C2$ *hold, then* $\det V > 0$.

*Proof.* We first observe that we may assume $V$ to be at least tri-diagonal, i.e.,

$$\begin{aligned} &\alpha_{j_{r+1}}(i_r) > 0 \qquad \text{for} \quad r = 1, 2, ..., m-1; \\ &\alpha_{j_{r-1}}(i_r) > 0 \qquad \text{for} \quad r = 2, 3, ..., m. \end{aligned} \tag{11}$$

For suppose, for example, that $\alpha_{j_{q+1}}(i_q) = 0$ for some $q$. Because of the support properties of discrete B-splines (Lemma 4), and since the diagonal is assumed to be positive, we see that in this case $V$ is block lower triangular. Its determinant is therefore given by the product of the determinants of the diagonal blocks,

$$\det V = \det A \begin{bmatrix} i_1, ..., i_m \\ j_1, ..., j_m \end{bmatrix} = \det A \begin{bmatrix} i_1, ..., i_q \\ j_1, ..., j_q \end{bmatrix} \det A \begin{bmatrix} i_{q+1}, ..., i_m \\ j_{q+1}, ..., j_m \end{bmatrix},$$

and $V$ is nonsingular if and only if both of the diagonal blocks are nonsingular. The case that some $\alpha_{j_{q-1}}(i_q)=0$ leads to a block upper triangular matrix and can be treated similarly.

In this way we can reduce the dimension of the matrix under consideration until it satisfies (11), or we end up with $m=1$, in which case the nonsingularity is trivial. In the rest of the proof we can therefore assume that (11) holds.

The proof of Lemma 10 is again by induction on the number of new knots, so let $\mathbf{s}$ be the sequence of new knots, as in the proof of Lemma 9. If $\#\mathbf{s}=0$, then $\boldsymbol{\tau}=\mathbf{t}$ so that we must have $V=I$, the identity matrix of order $m$ (here we do not need to use (11)).

If $\#\mathbf{s}=1$, then by Lemma 3 and (11), we see that we can assume $m=1$, in which case the nonsingularity is trivial.

For the general case, let $t_z$ be the first knot in $\mathbf{s}$ and form the knot vector $\boldsymbol{\rho}$ by dropping $t_z$ from $\mathbf{t}$. As in the proof of Lemma 9, we use Lemma 5 and find

$$\det V=\det A_{\boldsymbol{\tau},\mathbf{t}}\begin{bmatrix} i_1, ..., i_m \\ j_1, ..., j_m \end{bmatrix}$$

$$=\sum_{\xi_1<\cdots<\xi_m} \det A_{\boldsymbol{\rho},\mathbf{t}}\begin{bmatrix} i_1, ..., i_m \\ \xi_1, ..., \xi_m \end{bmatrix} \det A_{\boldsymbol{\tau},\boldsymbol{\rho}}\begin{bmatrix} \xi_1, ..., \xi_m \\ j_1, ..., j_m \end{bmatrix}. \tag{12}$$

Since we know that all minors are nonnegative, it is sufficient to exhibit a set of $\xi_r$'s such that the corresponding product in the sum in (12) is positive. As in the proof of Lemma 9, let us introduce the matrices

$$B_{\boldsymbol{\xi}}=A_{\boldsymbol{\rho},\mathbf{t}}\begin{bmatrix} i_1, ..., i_m \\ \xi_1, ..., \xi_m \end{bmatrix}, \qquad C_{\boldsymbol{\xi}}=A_{\boldsymbol{\tau},\boldsymbol{\rho}}\begin{bmatrix} \xi_1, ..., \xi_m \\ j_1, ..., j_m \end{bmatrix}.$$

From Lemma 3 we see that in order to have $\det B_{\boldsymbol{\xi}}>0$, we must have $\xi_r=i_r-1$ for all $r$ such that $t_{i_r}\geqslant t_z$. Also $\xi_r=i_r$ for all $r$ with $t_{i_r+k}\leqslant t_z$. In other words, if we define $l$ and $u$ by $l=z+r_{\mathbf{t}}(z)-k$ and $u=z-l_{\mathbf{t}}(z)$, which by (7) is equivalent to

$$t_z=t_{l+k}<t_{l+k+1}, \qquad t_{u-1}<t_u=t_z,$$

then we must have $\xi_r=i_r$ for $i_r\leqslant l$ and $\xi_r=i_r-1$ for $i_r\geqslant u$. Thus, we see that if $i_{r_0+r}=l+r$ for $r=0, 1, ..., u-l$ for some $r_0$, then there is no increasing set of integers $(\xi_r)$ such that $\det B_{\boldsymbol{\xi}}>0$. This is because we must have $\xi_{r_0}=i_{r_0}$ and $\xi_{r_0+u-l}=i_{r_0+u-l}-1$, and, hence, $\xi_{r_0+u-l}-\xi_{r_0}=u-l-1$ so that $\boldsymbol{\xi}$ cannot be strictly increasing. In order to obtain $\det B_{\boldsymbol{\xi}}>0$, there must, therefore, be gaps in the sequence $(i_r)$ between $l$ and $u$. These gaps are given by the set

$$G(l, u)=\{\, j \mid l\leqslant j\leqslant u \text{ and } i_r\neq j \text{ for } r=1, 2, ..., m\}. \tag{13}$$

If $G(l, u)$ is nonempty, let $p$ be the first gap,

$$p = \min G(l, u), \tag{14}$$

and define the $\xi$ sequence by

$$\xi_r = \begin{cases} i_r & \text{for} \quad i_r < p; \\ i_r - 1 & \text{for} \quad i_r > p. \end{cases} \tag{15}$$

Then we know from Lemma 3 that $\det B_\xi > 0$, and by (11) that the diagonal of $C_\xi$ is positive. We therefore only have to check that $C2$ is satisfied by $C_\xi$ since then we have $\det C_\xi > 0$ by induction and, therefore, at least one positive term in the sum in (12).

Let us first show that when the conditions of Theorem 6 are satisfied, then $G(l, u)$ is nonempty so that $p$ as given by (14) is well defined. If there is a $q$ such that $i_q = u$, then $f_q = 0$ and $C2$ states that

$$i_{q-d_q} < i_q - d_q. \tag{16}$$

But note that $i_q - d_q + k = i_q + r_{\mathbf{t}}(i_q) = l + k$ so that $i_q - d_q = l$. The condition (16) therefore tells us that $i_{q-d_q} < l$ so that $G(l, u)$ must contain at least one element. On the other hand, if there is no $q$ such that $i_q = u$, then $u \in G(l, u)$. For the rest of the proof we only consider $\xi$ as given by (15).

Let us next show that $C2$ is satisfied by $C_\xi$ for the values of $q$ such that $t_{i_q} = t_z$. If $f_q = 0$, $C2$ will not apply to $\xi_q$ in $\boldsymbol{\rho}$ since $f_q = 0$ means that either is $m_{\boldsymbol{\tau}}(t_z) = m_{\mathbf{t}}(t_z) - 1 = m_{\boldsymbol{\rho}}(t_z)$ or $i_q = u$. In both cases we have $m_{\boldsymbol{\rho}}(p_{\xi_q}) = m_{\boldsymbol{\tau}}(\rho_{\xi_q})$.

If $f_q > 0$, we note that $t_{i_q} = \rho_{\xi_q}$ and $r_{\mathbf{t}}(i_q) = r_{\boldsymbol{\rho}}(\xi_q)$, so that $\tilde{d}_q = d_q$, where $\tilde{d}_q$ is the value of the parameter $d_q$ for $C_\xi$. On the other hand, we have $l_{\boldsymbol{\rho}}(\xi_q) = l_{\mathbf{t}}(i_q) - 1$, and $m_{\boldsymbol{\rho}}(\xi_q) = m_{\mathbf{t}}(i_q) - 1$, so that $\tilde{f}_q = f_q - 1$, where $\tilde{f}_q$ is the value of $f_q$ for $C_\xi$. Since we also have $\xi_{q-d_q} = i_{q-d_q}$ (the inequality $i_{q-d_q} < l$ is always true) and $\xi_q = i_q - 1$, we find

$$\xi_{q-d_q} = i_{q-d_q} < i_q - d_q - f_q = \xi_q - d_q - (f_q - 1) = \xi_q - \tilde{d}_q - \tilde{f}_q.$$

Hence, $C2$ is satisfied by $\xi_q$ also in this case.

Finally, we need to show that $C2$ is also satisfied by $C_\xi$ for the values of $q$ for which $t_{i_q} > t_z$. So suppose that $t_{i_q} = t_y > t_z$ and $m_{\mathbf{t}}(t_y) < m_{\boldsymbol{\tau}}(t_y)$. By (15) we have $\xi_q = i_q - 1$, and by assumption we also have

$$i_{q-d_q} < i_q - d_q - f_q. \tag{17}$$

Since $\boldsymbol{\rho}$ is formed by dropping one knot at $t_z$ from $\mathbf{t}$, we see that $\tilde{f}_q = f_q$ and $\tilde{d}_q = d_q$. Therefore, if $i_{q-d_q} > p$ (recall that $p$ is the first gap in $G(l, u)$), we have $\xi_{q-d_q} = i_{q-d_q} - 1$ and, therefore,

$$\xi_{q-d_q} = i_{q-d_q} - 1 < i_q - d_q - f_q - 1 = \xi_q - f_q - d_q,$$

so that $C2$ is satisfied.

The only remaining difficulty is the case $i_{q-d_q} < p$. If in addition $\xi_{q-d_q} = i_{q-d_q} = i_q - d_q - f_q - 1$, then it is not a priori obvious that $C2$ is satisfied by $\xi_q$ on $\boldsymbol{\rho}$ since the minimum $f_q + 1$ gaps between $i_{q-d_q}$ and $i_q$ guaranteed by (17) could be reduced to only $f_q$ gaps between $\xi_{q-d_q}$ and $\xi_q$. However, we shall prove below that $i_q \geqslant i_{q-d_q} + d_q + f_q + 2$; i.e., there are at least $f_q + 2$ gaps.

The hypothesis is that $i_{q-d_q} < p$; then we also have $\xi_{q-d_q} = i_{q-d_q}$ and $\xi_q = i_q - 1$. If now $r_{\mathbf{t}}(i_q) > i_{q-d_q} - l$, then $d_q = k - r_{\mathbf{t}}(i_q) \leqslant k + l - 1 - i_{q-d_q}$. Using the fact that $t_{i_q} > t_z$, it now follows, as required, that

$$i_q \geqslant (k + l) + l_{\mathbf{t}}(i_q) + 1 \geqslant i_{q-d_q} + d_q + f_q + 2.$$

If instead $r_{\mathbf{t}}(i_q) \leqslant i_{q-d_q} - l$, we exploit the fact that there is an $r$ with $1 \leqslant r \leqslant q - d_q$ such that $i_r = l$ (this is true since $i_{q-d_q} < p$). Since there is no gap between $l$ and $i_{q-d_q}$, we have

$$i_{q-d_q} - l = q - d_q - r, \tag{18}$$

which means that $q \geqslant r + d_q + r_{\mathbf{t}}(i_q) = r + k$. But since $\alpha_{j_r}(i_r) > 0$ and there is a new knot at $t_z$, we know from Lemma 4 that $\tau_{j_{r+k}} \geqslant \tau_{j_r + k} > t_z$. Lemma 4, combined with the fact that $\alpha_{j_q}(i_q) > 0$, also tells us that either is $\tau_{j_q} < t_{i_q}$ or $\tau_{j_q} = t_{i_q}$ and $r_{\mathbf{t}}(i_q) \leqslant r_\tau(j_q)$. In the former case ($\tau_{j_q} < t_{i_q}$) we have $t_z < \tau_{j_{k+r}} \leqslant \tau_{j_q} < t_y = t_{i_q}$ since $q \geqslant k + r$. Hence, there are at least $q - k - r + 1$ knots of $\boldsymbol{\tau}$ in the interval $(t_z, t_y)$. It follows that

$$i_q \geqslant (l + k) + (q - k - r + 1) + (f_q + 1) = q + l - r + f_q + 2,$$

From (18) we then obtain $i_q \geqslant i_{q-d_q} + d_q + f_q + 2$. In the latter case ($\tau_{j_q} = t_{i_q}$), the number of knots of $\boldsymbol{\tau}$ in $(t_z, t_y)$ is at least $q - k - r + 1 - l_\tau(j_q) - 1$. Hence,

$$\begin{aligned} i_q &\geqslant (k + l) + (q - k - r + 1 - l_\tau(j_q) - 1) + l_{\mathbf{t}}(i_q) + 1 \\ &= q + l - r + 1 + l_{\mathbf{t}}(i_q) - l_\tau(j_q). \end{aligned}$$

But the inequality $r_{\mathbf{t}}(i_q) \leqslant r_{\boldsymbol{\tau}}(j_q)$ means that $l_{\mathbf{t}}(i_q) - l_{\boldsymbol{\tau}}(j_q) \geqslant m_{\mathbf{t}}(t_y) - m_{\boldsymbol{\tau}}(t_y) \geqslant f_q + 1$. Therefore $i_q \geqslant q + l - r + f_q + 2 = i_{q-d_q} + d_q + f_q + 2$ in this case too. ∎

This sequence of lemmas also finishes the proof of Theorem 6.

## 4. An Application of Theorem 6

Many properties of B-splines are special cases of discrete B-spline properties. It is therefore natural to try to obtain the total positivity of the B-spline collocation matrix from Theorem 6, see Jia's Remark 3 in [2]. This is indeed possible and quite simple, but before we consider this, let us introduce some notation. An alternative derivation, also based on knot insertion, can be found in [1].

Given $n$ real function $g_1, g_2, ..., g_n$, and $n$ points $(x_i)_{i=1}^n$ in their common domain, we denote the collocation matrix with elements $(g_j(x_i))$ by

$$M\begin{pmatrix} x_1, ..., x_n \\ g_1, ..., g_n \end{pmatrix}, \tag{19}$$

and its determinant by

$$D\begin{pmatrix} x_1, ..., x_n \\ g_1, ..., g_n \end{pmatrix}. \tag{20}$$

Recall that the interpolation problem

$$\sum_{j=1}^{n} c_j g_j(x_i) = y_i \qquad \text{for} \quad i = 1, 2, ..., n,$$

has a unique solution if and only if the determinant in (20) is nonzero. If the functions $\{g_j\}$ are B-splines, the following theorem provides important information about the matrix in (19).

Theorem 11. *Let $\boldsymbol{\tau}$ be a knot vector with $\tau_j < \tau_{j+k}$ for all $j$, let $\{B_j\}$ be the corresponding B-splines, and let $(x_i)_{i=1}^n$ be $n$ distinct real numbers. Then*

$$D\begin{pmatrix} x_1, ..., x_n \\ B_{j_1}, ..., B_{j_n} \end{pmatrix} \geqslant 0,$$

*with strict inequality if and only if $B_{j_r}(x_r) > 0$ for $r = 1, 2, ..., n$.*

*Proof.* Let $\mathbf{t}$ be a knot vector that contains $\boldsymbol{\tau}$ and in which each number $x_r$ occurs $k$ times. From the definition of discrete B-splines (1), we see that if $t_{i_r+1} = t_{i_r+k-1} = x_r < t_{i_r+k}$ (the last inequality is required when our B-splines are right continuous), then we have $B_{j,k,\boldsymbol{\tau}}(x_r) = \alpha_{j,k,\boldsymbol{\tau},\mathbf{t}}(i_r)$. We therefore have

$$M\begin{pmatrix} x_1, ..., x_n \\ B_{j_1}, ..., B_{j_n} \end{pmatrix} = A_{\boldsymbol{\tau},\mathbf{t}}\begin{pmatrix} i_1, ..., i_n \\ j_1, ..., j_n \end{pmatrix},$$

and because of $C1$, we need the diagonal to be positive to have a positive determinant. We also see that by letting $\mathbf{t}$ be a knot vector with sufficiently many knots between the points $(x_r)$, the condition $C2$ will be automatically satisfied for our particular choice of the $(i_r)$. The result therefore follows from Theorem 6. ∎

In the case where $j_r = r$ for all $r$, Theorem 11 is usually referred to as the Schoenberg–Whitney theorem. It tells us that if the spline interpolation problem

$$\sum_{i=1}^{n} c_i B_{i,k}(x_r) = y_r \qquad \text{for} \quad r = 1, 2, ..., n,$$

is to have a unique solution, then the interpolation point $x_r$ must be inside the support of $B_{r,k}$.

A familiar extension of this result ensures unicity event when the interpolation points are allowed to coalesce; i.e., derivatives are interpolated as well. As we shall see, this result is also an easy consequence of Theorem 6. If we have coalescent interpolation points $(x_r)$, the interpolation problem is

$$\sum_{i=1}^{n} c_i D^{\mu_r} B_{i,k}(x_r) = y_r \qquad \text{for} \quad r = 1, 2, ..., n,$$

where $\mu_r$ is the number of integers $j$ that satisfy $j < r$ and $x_j = x_r$; in other words, $\mu_r = l_{\mathbf{x}}(r)$. Since we are considering splines of order $k$, we must of course have $\mu_r \leqslant k-1$ for all $r$. If this problem is to have a unique solution, the coefficient matrix

$$C = (D^{\mu_i} B_{j,k}(x_i))_{i,j=1}^{n}$$

must be nonsingular.

Since the derivatives of a B-spline may be negative at a point, the matrix $C$ is not necessarily totally positive. On the other hand, if the submatrix $V$ is formed by rows $(i_r)_{r=l}^{m}$ and columns $(j_r)_{r=1}^{m}$, without leaving "gaps" in the derivatives at a point, then we do have $\det V \geqslant 0$.

THEOREM 12. *Let $\boldsymbol{\tau}$ be a knot vector with $\tau_j < \tau_{j+k}$, and let $C$ be the matrix*

$$C = (D^{\mu_i} B_{j,k}(x_i))_{i,j=1}^{n}, \tag{21}$$

*where $\mu_i \neq \#\{j \mid j < i$ and $x_j = x_i\}$. Let $V$ be the submatrix of $C$ formed by taking rows $i_1 < i_2 < \cdots < i_m$ and columns $j_1 < j_2 < \cdots < j_m$ that satisfy the conditions*

$$i_{r-1} < i_r - 1 \qquad \textit{implies} \quad x_{i_r - 1} < x_{i_r} \quad \text{for } r = 2, 3, ..., m. \tag{22}$$

*Then*

$$\det V \geqslant 0,$$

*with strict inequality if and only if for each $r \in \{1, 2, ..., m\}$ one of the following two conditions are satisfied,*

(i) $B_{j_r, k}(x_{i_r}) > 0$,

(ii) $\tau_{j_r} = x_{i_r}$ *and* $D^{\mu_r} B_{j_r}(x_{i_r}) \neq 0$.

*Proof.* The idea behind the proof is again to identify $V$ with a submatrix of a discrete B-spline collocation matrix. Set $z_r = x_{i_r}$ for $r = 1, 2, ..., m$. Then we have

$$V = (D^{l_{\mathbf{z}}(r)} B_{j_q, k}(z_r))_{r, q=1}^{m}.$$

Form the knot vectors $\mathbf{t}$ as in the proof of Theorem 11; i.e., let $\mathbf{t}$ contain at least both $\boldsymbol{\tau}$ and the interpolation points $(z_r)$ as subsequences, but such that all distinct knots in $\mathbf{t}$ occur exactly $k$ times (we shall see below that it may be convenient to let $\mathbf{t}$ contain even more knots). Let $a$ be an interpolation point and define $s$ and $p$ by

$$z_{s-1} < a = z_s = \cdots z_{s+p} < z_{s+p+1},$$

so that we interpolate up to $p$ th order derivatives at $a$. Define the integer $\xi_s$ by

$$t_{\xi_s - 1} < a = t_{\xi_s} = \cdots t_{\xi_s + k - 1} < t_{\xi_s + k}.$$

From Lemma 1 we then have

$$D^l B_{j, \boldsymbol{\tau}}(a) = \frac{(k-1) \cdots (k-l)}{h^l} \nabla^l \alpha_{j, k, \boldsymbol{\tau}, \mathbf{t}}(\xi_s + l) \qquad \text{for} \quad l = 0, 1, ..., p,$$

where $h = t_{\xi_s + k} - t_{\xi_s} = z_{s+p+1} - z_s$, and $\nabla \alpha_{j,k}(i) = \alpha_{j,k}(i) - \alpha_{j,k}(i-1)$. If we denote row $s$ of $V$ by $B_{j_r}(z_s)$, this means that

$$\det V = \det \begin{bmatrix} B_{j_r}(z_1) \\ \vdots \\ B_{j_r}(z_s) \\ DB_{j_r}(z_{s+1}) \\ \vdots \\ D^p B_{j_r}(z_{s+p}) \\ \vdots \\ D^{\mu_m} B_{j_r}(z_m) \end{bmatrix}$$

$$= \frac{(k-1)^p\,(k-2)^{p-1}\cdots(k-p)}{h^{p(p+1)/2}} \det \begin{bmatrix} B_{j_r}(z_1) \\ \vdots \\ \alpha_{j_r}(\xi_s) \\ \nabla\alpha_{j_r}(\xi_s+1) \\ \vdots \\ \nabla^p \alpha_{j_r}(\xi_s+p) \\ \vdots \\ D^{\mu_m} B_{j_r}(z_m) \end{bmatrix}. \tag{23}$$

Now, we have $\nabla^l\, d(i) = \sum_{j=0}^{l} (-1)^j \binom{l}{j}\, d(i-j)$. Therefore, if we set $\xi_{s+l} = \xi_s + l$ for $l = 1, ..., p$, then by elementary properties of the determinant, the right-hand side of (23) reduces to

$$\det V = \frac{(k-1)^p\,(k-2)^{p-1}\cdots(k-p)}{h^{p(p+1)/2}} \det \begin{bmatrix} B_{j_r}(z_1) \\ \vdots \\ \alpha_{j_r}(\xi_s) \\ \alpha_{j_r}(\xi_{s+1}) \\ \vdots \\ \alpha_{j_r}(\xi_{s+p}) \\ \vdots \\ B_{j_r}(z_m) \end{bmatrix}. \tag{23}$$

If we do this for all the distinct interpolation points, we see that

$$\det V = c \det A \begin{bmatrix} \xi_1, ..., \xi_m \\ j_1, ..., j_m \end{bmatrix},$$

where $A = A_{\boldsymbol{\tau}, \mathbf{t}}$ is a discrete B-spline collocation matrix, and $c$ some positive constant. Hence, by Theorem 6, the matrix $V$ is nonsingular if and only if $\alpha_{j_r}(\xi_r) > 0$ for $r = 1, ..., m$ (we can always satisfy $C2$ of Theorem 6 by letting the knot spacing in $\mathbf{t}$ be small). From Lemma 4 we see that $\alpha_{j_r}(\xi_r) > 0$ if and only if either (i) $\tau_{j_r} < t_{\xi_r} < \tau_{j_r+k}$, or (ii) $\tau_{j_r} = t_{\xi_r}$ and $r_\tau(j_r) \geqslant r_{\mathbf{t}}(\xi_r)$ (the situation $\alpha_{j_r}(\xi_r) > 0$ and $\tau_{j_r+k} = t_{\xi_r+k}$ and $l_{\mathbf{t}}(\xi_r + k) \leqslant l_{\boldsymbol{\tau}}(j_r + k)$ can always be avoided by choosing $\mathbf{t}$ appropriately). The former condition is equivalent to $B_{j_r, \boldsymbol{\tau}}(z_r) > 0$. The latter condition is equivalent to $\tau_{j_r} = z_r$ and $r_{\boldsymbol{\tau}}(j_r) \geqslant k - 1 - l_{\mathbf{z}}(r)$ since $l_{\mathbf{z}}(r) = l_{\mathbf{t}}(\xi_r)$ and $l_{\mathbf{t}}(\xi_r) + r_{\mathbf{t}}(\xi_r) = k - 1$. But this is in turn equivalent to $\tau_{j_r} = z_r$ and $D^{(l_{\mathbf{z}}(r))} B_{j_r}(z_r) \neq 0$ from which the result follows. ∎

## Acknowledgments

I thank one of the referees for a very thorough review and constructive comments that helped improve the presentation of this paper. In particular, the present form of the last part of the proof of Lemma 10 was suggested by the referee.

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